

# A QUENCHED LARGE DEVIATION PRINCIPLE AND A PARISI FORMULA FOR A PERCEPTRON VERSION OF THE GREM.

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**ABSTRACT.** We introduce a perceptron version of the Generalized Random Energy Model, and prove a quenched Sanov type large deviation principle for the empirical distribution of the random energies. The dual of the rate function has a representation through a variational formula which is closely related to the Parisi variational formula for the SK-model.

*Dedicated to Jürgen Gärtner on the occasion of his 60th birthday.*

## 1. INTRODUCTION

There has been important progress in the mathematical study of mean field spin glasses over the last 10 years. By results of Guerra [10] and Talagrand [14], the free energy of the Sherrington-Kirkpatrick model is known to be given by the formula predicted by Parisi [9]. Furthermore, the description of the *high* temperature is remarkably accurate, see [13] and references therein. On the other hand, results for the Gibbs measure at *low* temperature are more scarce and are restricted to models with a simpler structure, like Derrida's generalized random energy model, the GREM, [5] and [8], the nonhierarchical GREMs [2] and the  $p$ -spin model with large  $p$  [13]. To put on rigorous ground the full Parisi picture remains a major challenge, and even more so in view of its alleged universality, at least for mean-field models.

We introduce here a model which hopefully sheds some new light on the issue.

In this paper we derive the free energy, which can be analyzed by large deviation techniques. The limiting free energy turns out to be given by a Gibbs variational formula which can be linked to a Parisi-type formula by a duality principle, so that it becomes evident why an infimum appears in the latter. This duality also gives an interesting interpretation of the Parisi order parameter in terms of the sequence of inverse of temperatures associated to the extremal measures from the Gibbs variational principle.

In a forthcoming paper, we will give a full description of the Gibbs measure in the thermodynamic limit in terms of the Ruelle cascades.

## 2. A PERCEPTRON VERSION OF THE GREM

Let  $\{X_{\alpha,i}\}_{\alpha \in \Sigma_N, 1 \leq i \leq N}$ , be random variables which take values in a Polish space  $S$  equipped with the Borel  $\sigma$ -field  $\mathcal{S}$ , and defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We write  $\mathcal{M}_1^+(S)$  for the set of probability measures on  $(S, \mathcal{S})$ , which itself is a Polish space.  $\Sigma_N$  is exponential in size, typically  $|\Sigma_N| = 2^N$ . It is assumed that all  $X_{\alpha,i}$  have the same distribution  $\mu$ , and that for any

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fixed  $\alpha \in \Sigma_N$ , the collection  $\{X_{\alpha,i}\}_{1 \leq i \leq N}$  is independent. It is however not assumed that they are independent for different  $\alpha$ . The perceptron Hamiltonian is defined by

$$-H_{N,\omega}(\alpha) \stackrel{\text{def}}{=} \sum_{i=1}^N \phi(X_{\alpha,i}(\omega)), \quad (2.1)$$

where  $\phi : S \rightarrow \mathbb{R}$  is a measurable function. One may allow that the index set for  $i$  is rather  $\{1, \dots, [aN]\}$  with  $a$  some positive real number, but for convenience, we always stick to  $a = 1$  here. The case which is best investigated (see [13]) takes for  $\alpha$  spin sequences:  $\alpha = (\sigma_1, \dots, \sigma_N) \in \{-1, 1\}^N$ ,  $S = \mathbb{R}$ , and the  $X_{\alpha,i}$  are centered Gaussians with

$$\mathbb{E}(X_{\alpha,i} X_{\alpha',i'}) = \delta_{i,i'} \frac{1}{N} \sum_{j=1}^N \sigma_j \sigma'_j. \quad (2.2)$$

This is closely related to the SK-model, and is actually considerably more difficult. The model has been investigated by Talagrand [13], but a full Parisi formula for the free energy is lacking.

The Hamiltonian (2.1) can be written in terms of the empirical measure

$$L_{N,\alpha} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N \delta_{X_{\alpha,i}} \quad (2.3)$$

i.e.

$$-H_{N,\omega}(\alpha) = N \int \phi(x) L_{N,\alpha}(dx).$$

The quenched free energy is the almost sure limit of

$$\frac{1}{N} \log \sum_{\alpha} \exp[-H_{N,\omega}(\alpha)],$$

and it appears natural to ask if this free energy can be obtained by a quenched Sanov type large deviation principle for  $L_{N,\alpha}$  in the following form:

**Definition 2.1.** *We say that  $\{L_N\}$  satisfies a **quenched large deviation principle** (in short QLDP) with good rate function  $J : \mathcal{M}_1^+(S) \rightarrow [-\infty, \infty]$ , provided the level sets of  $J$  are compact, and for any weakly continuous bounded map  $\Phi : \mathcal{M}_1^+(S) \rightarrow \mathbb{R}$ , one has*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \sum_{\alpha \in \Sigma_N} \exp[N\Phi(L_{N,\alpha})] = \log 2 + \sup_{\nu \in \mathcal{M}_1^+(S)} [\Phi(\nu) - J(\mu)], \quad \mathbb{P}\text{-a.s.}$$

The annealed version of such a QLDP is just Sanov's theorem:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \log \sum_{\alpha} \mathbb{E} \exp[N\Phi(L_{N,\alpha})] &= \log 2 + \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \exp[N\Phi(L_{N,\alpha})] \\ &= \log 2 + \sup_{\nu} (\Phi(\nu) - H(\nu|\mu)) \end{aligned}$$

where  $H(\nu|\mu)$  is the usual relative entropy of  $\nu$  with respect to  $\mu$ , the latter being the distribution of the  $X_{\alpha,i}$ :

$$H(\nu|\mu) \stackrel{\text{def}}{=} \begin{cases} \int \log \frac{d\nu}{d\mu} d\nu & \text{if } \nu \ll \mu \\ \infty & \text{otherwise} \end{cases}.$$

There is no reason to believe that  $H(\nu|\mu) = J(\nu)$ .

**Conjecture 2.2.** *The empirical measures  $\{L_{N,\alpha}\}$  with (2.2) satisfy a QLDP.*

We don't know how this conjecture could be proved, nor do we have a clear picture what  $J$  should be in this case. The only support we have for the conjecture is that it is true in a perceptron version of the GREM, a model we are now going to describe.

For  $n \in \mathbb{N}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $1 \leq \alpha_k \leq 2^{\gamma_i N}$ ,  $\sum_k \gamma_k = 1$ , and  $1 \leq i \leq N$ , let

$$X_{\alpha,i} = (X_{\alpha_1,i}^1, X_{\alpha_1,\alpha_2,i}^2, \dots, X_{\alpha_1,\alpha_2,\dots,\alpha_n,i}^n)$$

where the  $X^j$  are independent, taking values in some Polish Space  $(S, \mathcal{S})$  with distribution  $\mu_j$ . For notational convenience, we assume that the  $\gamma_i N$  are all integers. Put

$$\Gamma_j \stackrel{\text{def}}{=} \sum_{k=1}^j \gamma_k.$$

We assume that all the variables in the bracket are independent. The  $X_{\alpha,i}$  take values in  $S^n$ . The distribution is

$$\mu \stackrel{\text{def}}{=} \mu_1 \otimes \dots \otimes \mu_n$$

The empirical measure  $L_{N,\alpha}$  is defined by (2.3) which is a random element in  $\mathcal{M}_1^+(S^n)$ .  $n$  is fixed in all we are doing.

Given a measure  $\nu \in \mathcal{M}_1^+(S^n)$ , and  $1 \leq j \leq n$ , we write  $\nu^{(j)}$  for its marginal on the first  $j$  coordinates. We define subsets  $\mathcal{R}_j$  of  $\mathcal{M}_1^+(S^n)$ ,  $1 \leq j \leq n$  by

$$\mathcal{R}_j \stackrel{\text{def}}{=} \left\{ \nu \in \mathcal{M}_1^+(S^n) : H\left(\nu^{(j)} \mid \mu^{(j)}\right) \leq \Gamma_j \log 2 \right\}.$$

We will also consider the sets

$$\mathcal{R}_j^- \stackrel{\text{def}}{=} \left\{ \nu \in \mathcal{M}_1^+(S^n) : H\left(\nu^{(j)} \mid \mu^{(j)}\right) = \Gamma_j \log 2 \right\}.$$

For  $\nu \in \mathcal{M}_1^+(S^n)$  let

$$J(\nu) = \begin{cases} H(\nu \mid \mu) & \text{if } \nu \in \bigcap_{j=1}^n \mathcal{R}_j \\ \infty & \text{otherwise} \end{cases}.$$

It is evident that  $J$  is convex and has compact level sets.

Our first main result is:

**Theorem 2.3.**  $\{L_{N,\alpha}\}$  satisfies a QLDP with rate function  $J$ .

For the rest of this section, we will focus on linear functionals,  $\Phi(\nu) = \int \phi(x) \nu(dx)$ , for a bounded continuous function  $\phi : S^n \rightarrow \mathbb{R}$ . For a probability measure  $\nu$  on  $S^n$ , we set

$$\text{Gibbs}(\phi, \nu) \stackrel{\text{def}}{=} \int \phi(x) \nu(dx) - H(\nu \mid \mu),$$

and define the Legendre transform of  $J$  by

$$J^*(\phi) \stackrel{\text{def}}{=} \sup_{\nu} \left[ \int \phi(x) \nu(dx) - J(\nu) \right] = \sup \left\{ \text{Gibbs}(\phi, \nu) : \nu \in \bigcap_{j=1}^n \mathcal{R}_j \right\}.$$

whenever the a.s.-limit exists. As a corollary of Theorem 2.3 we have

**Corollary 2.4.** Assume that  $\phi : S \rightarrow \mathbb{R}$  is bounded and continuous.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \sum_{\alpha} \exp \left[ \sum_{i=1}^N \phi(X_{\alpha,i}) \right] = J^*(\phi) + \log 2, \text{ a.s.}$$

We next discuss a dual representation of  $J^*(\phi)$ . Essentially, this comes up by investigating which measures solve the variational problem. Remark that without the restrictions  $\nu \in \bigcap_{j=1}^n \mathcal{R}_j$ , we would simply get

$$d\nu = \frac{e^\phi d\mu}{\int e^\phi d\mu}$$

as the maximizer.

Let  $\Delta$  be the set of sequences  $\mathbf{m} = (m_1, \dots, m_n)$  with  $0 < m_1 \leq m_2 \leq \dots \leq m_n \leq 1$ . For  $\mathbf{m} \in \Delta$ , and  $\phi : S^n \rightarrow \mathbb{R}$  bounded, we define recursively functions  $\phi_j$ ,  $0 \leq j \leq n$ ,  $\phi_j : S^j \rightarrow \mathbb{R}$ , by

$$\phi_n \stackrel{\text{def}}{=} \phi, \quad (2.4)$$

$$\phi_{j-1}(x_1, \dots, x_{j-1}) \stackrel{\text{def}}{=} \frac{1}{m_j} \log \int \exp[m_j \phi_j(x_1, \dots, x_{j-1}, x_j)] \mu_j(dx_j). \quad (2.5)$$

$\phi_0$  is just a real number, which we denote by  $\phi_0(\mathbf{m})$ .

Remark that if some of the  $m_i$  agree, say  $m_k = m_{k+1} = \dots = m_l$ ,  $k < l$ , then  $\phi_{k-1}$  is obtained from  $\phi_l$  by

$$\phi_{k-1}(x_1, \dots, x_{k-1}) = \frac{1}{m_k} \log \int \exp[m_k \phi_l(x_1, \dots, x_{k-1}, x_k, \dots, x_l)] \prod_{j=k}^l \mu_j(dx_j).$$

In particular, if all the  $m_i$  are 1, then

$$\phi_0 = \log \int \exp[\phi] d\mu.$$

This latter case corresponds to the “replica symmetric” situation. Put

$$\text{Parisi}(\mathbf{m}, \phi) \stackrel{\text{def}}{=} \sum_{i=1}^n \frac{\gamma_i \log 2}{m_i} + \phi_0(\mathbf{m}) - \log 2 \quad (2.6)$$

**Theorem 2.5.** *Assume that  $\phi : S \rightarrow \mathbb{R}$  is bounded and continuous. Then*

$$J^*(\phi) = \inf_{\mathbf{m} \in \Delta} \text{Parisi}(\mathbf{m}, \phi). \quad (2.7)$$

The expression for  $J^*(\phi)$  in this theorem is very similar to the Parisi formula for the SK-model. Essentially the only difference is the first summand which in the SK-case is a quadratic expression. In our case (in contrast to the still open situation in the SK-model), we can prove that the infimum is uniquely attained, as we will discuss below.

The derivation of the theorem from Corollary 2.4 is done by identifying first the possible maximizers in the variational formula for  $J^*(\phi)$ . They belong to a family of distributions, parametrized by  $\mathbf{m}$ . The maximizer inside this family is then obtained by minimizing  $\mathbf{m}$  according to (2.7), and one then identifies the two expressions. The procedure is quite standard in large deviation situations.

Two conventions:  $C$  stands for a generic positive constant, not necessarily the same at different occurrences. If there are inequalities stated between expressions containing  $N$ , it is tacitly assumed that they are valid maybe only for large enough  $N$ .

### 3. PROOFS

**3.1. The Gibbs variational principle: Proof of Theorem 2.3.** If  $A \in \mathcal{S}$ , we put  $H(A \mid \mu) \stackrel{\text{def}}{=} \inf_{\nu \in \mathcal{A}} H(\nu \mid \mu)$ . If  $S$  is a Polish Space, and  $\mathcal{S}$  its Borel  $\sigma$ -field, then it is well known that

$\nu \rightarrow H(\nu \mid \mu)$  is lower semicontinuous in the weak topology. This follows from the representation

$$H(\nu \mid \mu) = \sup_{u \in \mathcal{U}} \left[ \int u d\nu - \log \int e^u d\mu \right], \quad (3.1)$$

where  $\mathcal{U}$  is the set of bounded continuous functions  $S \rightarrow \mathbb{R}$ .

For  $(S, \mathcal{S}), (S', \mathcal{S}')$  two Polish Spaces, and  $\nu \in \mathcal{M}_1^+(S \times S')$ . If  $\mu \in \mathcal{M}_1^+(S)$ ,  $\mu' \in \mathcal{M}_1^+(S')$  we have,

$$H(\nu \mid \mu \otimes \mu') = H(\nu^{(1)} \mid \mu) + H(\nu \mid \nu^{(1)} \otimes \mu'), \quad (3.2)$$

where  $\nu^{(1)}$  is the first marginal of  $\nu$  on  $S$ .

**Lemma 3.1.**  *$H(\nu \mid \nu^{(1)} \otimes \mu')$  is a lower semicontinuous function of  $\nu$  in the weak topology.*

*Proof.* Applying (3.1) to

$$H(\nu \mid \nu^{(1)} \otimes \mu') = \sup_{u \in \mathcal{U}} \left[ \int u d\nu - \log \int e^u d(\nu^{(1)} \otimes \mu') \right],$$

where  $\mathcal{U}$  denotes the set of bounded continuous functions  $S \times S' \rightarrow \mathbb{R}$ . For any fixed  $u \in \mathcal{U}$ , both functions  $\nu \rightarrow \int u d\nu$  and  $\nu \rightarrow \log \int e^u d(\nu^{(1)} \otimes \mu')$  are continuous, and from this the desired semicontinuity property follows.  $\square$

We will need the following “relative” version of Sanov’s theorem. Consider three independent sequences of i.i.d. random variables  $(X_i), (Y_i), (Z_i)$ , taking values in three Polish spaces  $S, S', S''$ , and with laws  $\mu, \mu', \mu''$ . We consider the empirical processes

$$L_N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N \delta_{(X_i, Y_i)}, \quad R_N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N \delta_{(X_i, Z_i)}.$$

The pair  $(L_N, R_N)$  takes values in  $\mathcal{M}_1^+(S \times S') \times \mathcal{M}_1^+(S \times S'')$ .

**Lemma 3.2.** *The sequence  $(L_N, R_N)$  satisfies a LDP with rate function*

$$J(\nu, \theta) = \begin{cases} H(\nu^{(1)} \mid \mu) + H(\nu \mid \nu^{(1)} \otimes \mu') + H(\theta \mid \theta^{(1)} \otimes \mu''), & \text{if } \nu^{(1)} = \theta^{(1)} \\ \infty & \text{otherwise.} \end{cases}$$

*Proof.* We apply the Sanov theorem to the empirical measure

$$M_N = \frac{1}{N} \sum_{i=1}^N \delta_{(X_i, Y_i, Z_i)} \in \mathcal{M}_1^+(S \times S' \times S'').$$

We use the two natural projections  $p : S \times S' \times S'' \rightarrow S \times S'$  and  $q : S \times S' \times S'' \rightarrow S \times S''$ . Then  $(L_N, R_N) = M_N(p, q)^{-1}$ , and by continuous projection, we get that  $(L_N, R_N)$  satisfies a good LDP with rate function

$$J'(\nu, \theta) = \inf \{ H(\rho \mid \mu \otimes \mu' \otimes \mu'') : \rho p^{-1} = \nu, \rho q^{-1} = \theta \}.$$

It only remains to identify this rate function with the function  $J$  given above.

Clearly  $J'(\nu, \theta) = \infty$  if  $\nu^{(1)} \neq \theta^{(1)}$ . Therefore, assume  $\nu^{(1)} = \theta^{(1)}$ . If we define  $\hat{\rho}(\nu, \theta) \in \mathcal{M}_1^+(S \times S' \times S'')$  to have marginal  $\nu^{(1)} = \theta^{(1)}$  on  $S$ , and the conditional distribution on  $S' \times S''$  given the first projection is the product of the conditional distributions of  $\nu$  and  $\theta$ , then applying twice (3.2), we get

$$H(\hat{\rho} \mid \mu \otimes \mu' \otimes \mu'') = H(\nu^{(1)} \mid \mu) + H(\nu \mid \nu^{(1)} \otimes \mu') + H(\theta \mid \theta^{(1)} \otimes \mu''),$$

and therefore  $J \geq J'$ .

To prove the other inequality, consider any  $\rho$  satisfying  $\rho p^{-1} = \nu, \rho q^{-1} = \theta$ . We want to show that  $J(\nu, \theta) \leq H(\rho \mid \mu \otimes \mu' \otimes \mu'')$ . For that, we can assume that the right hand side is finite. Then

$$H(\rho \mid \mu \otimes \mu' \otimes \mu'') = H(\rho \mid \hat{\rho}(\nu, \theta)) + \int d\rho \log \frac{d\hat{\rho}(\nu, \theta)}{d(\mu \otimes \mu' \otimes \mu'')}.$$

The first summand is  $\geq 0$ , and the second equals

$$\int d\hat{\rho}(\nu, \theta) \log \frac{d\hat{\rho}(\nu, \theta)}{d(\mu \otimes \mu' \otimes \mu'')} = J(\nu, \theta).$$

So, we have proved that

$$J(\nu, \theta) \leq H(\rho \mid \mu \otimes \mu' \otimes \mu''),$$

for any  $\rho$  satisfying  $\rho p^{-1} = \nu, \rho q^{-1} = \theta$ .  $\square$

We now step back to the setting of Theorem 2.3: For  $j = 1, \dots, n$ , we have sequences  $\{X_{\alpha_1, \dots, \alpha_j, i}^j\}$  of independent random variables with distribution  $\mu_j$ . We emphasize that henceforth  $\mu = \mu_1 \otimes \dots \otimes \mu_n$  and  $\mu^{(j)}$  will denote the marginal on the first  $k$  components. Moreover, for  $\alpha = (\alpha_1, \dots, \alpha_n)$ , we write  $\alpha^{(j)} = (\alpha_1, \dots, \alpha_j)$  and set

$$L_{N, \alpha^{(j)}}^{(j)} = \frac{1}{N} \sum_{i=1}^N \delta_{(X_{\alpha_1, i}^1, X_{\alpha_1, \alpha_2, i}^2, \dots, X_{\alpha_1, \dots, \alpha_j, i}^j)},$$

for  $j \leq n$ , which is the marginal of  $L_{N, \alpha}$  on  $S^j$ . With the notation

$$\begin{aligned} X_{\alpha, i}^{(j)} &\stackrel{\text{def}}{=} (X_{\alpha_1, i}^1, \dots, X_{\alpha_1, \dots, \alpha_j, i}^j), \\ \hat{X}_{\alpha, i}^{(j)} &\stackrel{\text{def}}{=} (X_{\alpha_1, \dots, \alpha_{j+1}, i}^{j+1}, \dots, X_{\alpha_1, \dots, \alpha_n, i}^n), \end{aligned}$$

we can write

$$L_{N, \alpha} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^N \delta_{(X_{\alpha, i}^{(j)}, \hat{X}_{\alpha, i}^{(j)})}. \quad (3.3)$$

For  $A \subset \mathcal{M}_1^+(S^n)$  we put  $M_N(A) \stackrel{\text{def}}{=} \#\{\alpha : L_{N, \alpha} \in A\}$ .

**Lemma 3.3.** *Assume  $\nu \in \mathcal{M}_1^+(S^n)$  satisfies  $H(\nu \mid \mu) < \infty$ , and let  $V$  be an open neighborhood of  $\nu$ , and  $\varepsilon > 0$ . Then there exists an open neighborhood  $U$  of  $\nu$ ,  $U \subset V$ , and  $\delta > 0$  such that*

$$\mathbb{P}\left[M_N(U) \geq \exp[N(\log 2 - H(\nu \mid \mu) + \varepsilon)]\right] \leq e^{-\delta N}.$$

*Proof.* If  $B_r(\nu)$  denotes the open  $r$ -ball around  $\nu$  in one of the standard metrics, e.g. the Prohorov metric, then by the semicontinuity property of the relative entropy, one has

$$H(B_r(\nu) \mid \mu) \uparrow H(\nu \mid \mu)$$

as  $r \downarrow 0$ . We can choose a sequence  $r_k > 0, r_k \downarrow 0$  with  $H(B_{r_k}(\nu) \mid \mu) = H(\text{cl}(B_{r_k}(\nu)) \mid \mu) \uparrow H(\nu \mid \mu)$ . Given  $\varepsilon > 0$ , and  $V$ , we can find  $k$  such that

$$H(B_{r_k}(\nu) \mid \mu) = H(\text{cl}(B_{r_k}(\nu)) \mid \mu) \geq H(\nu \mid \mu) - \varepsilon/4$$

and  $B_{r_k}(\nu) \subset V$ . By Sanov's theorem we therefore get

$$\mathbb{P}\left[L_{N, \alpha} \in B_{r_k}(\nu)\right] \leq \exp[N(-H(\nu \mid \mu) + \varepsilon/2)],$$

and therefore

$$\mathbb{E}\left[M_N(B_{r_k}(\nu))\right] \leq \exp[N(\log 2 - H(\nu \mid \mu) + \varepsilon/2)].$$

By the Markov inequality, the claim follows by taking  $\delta = \varepsilon/3$ .  $\square$

**Lemma 3.4.** *Assume  $\nu \in \mathcal{M}_1^+(S^n)$  satisfies  $H(\nu^{(j)} | \mu^{(j)}) > \Gamma_j \log 2$  for some  $j \leq n$ , and let  $V$  be an open neighborhood of  $\nu$ . Then there is an open neighborhood  $U$  of  $\nu$ ,  $U \subset V$  and  $\delta > 0$  such that*

$$\mathbb{P}[M_N(U) \neq 0] \leq e^{-\delta N}$$

for large enough  $N$ .

*Proof.* As in the previous lemma, we choose a neighborhood  $U'$  of  $\nu^{(j)}$  in  $S^j$  such that  $H(\text{cl}(U') | \mu^{(j)}) = H(U' | \mu^{(j)}) > \Gamma_j \log 2 + \eta$ , for some  $\eta > 0$ . Then we put

$$U \stackrel{\text{def}}{=} \left\{ \nu \in \mathcal{M}_1^+(S^n) : \nu \in V, \nu^{(j)} \in U' \right\}.$$

If  $L_{N,\alpha} \in U$  then  $L_{N,\alpha}^{(j)} \in U'$ ,

$$\begin{aligned} \mathbb{P}[\exists \alpha : L_{N,\alpha} \in U] &\leq \mathbb{P}[\exists \alpha : L_{N,\alpha}^{(j)} \in U'] \\ &\leq 2^{\Gamma_j N} \mathbb{P}[L_{N,\alpha}^{(j)} \in U'] \\ &\leq 2^{\Gamma_j N} \exp \left[ -NH \left( \text{cl}(U') | \mu^{(j)} \right) + N\eta/2 \right] \\ &\leq 2^{\Gamma_j N} \exp [-N\Gamma_j \log 2 - N\eta/2] = e^{-N\eta/2}. \end{aligned}$$

This proves the claim.  $\square$

**Lemma 3.5.** *Assume that  $\nu \in \mathcal{M}_1^+(S^n)$  satisfies  $H(\nu^{(j)} | \mu^{(j)}) < \Gamma_j \log 2$  for all  $j$ , and let  $V$  be an open neighborhood of  $\nu$ , and  $\varepsilon > 0$ . Then there exists an open neighborhood  $U$  of  $\nu$ ,  $U \subset V$ , and a  $\delta > 0$  such that*

$$\mathbb{P} \left[ M_N(U) \leq \exp [N(\log 2 - H(\nu | \mu) - \varepsilon)] \right] \leq e^{-\delta N}.$$

*Proof.* We claim that we can find  $U$  as required, and some  $\delta > 0$ , such that

$$\text{var} [M_N(U)] \leq e^{-2N\delta} \{ \mathbb{E} [M_N(U)] \}^2 \quad (3.4)$$

From this estimate, we easily get the claim: From Sanov's theorem, we have for any  $\chi > 0$

$$\mathbb{E} M_N(U) = 2^N \mathbb{P}(L_{N,\alpha} \in U) \geq \exp [N(\log 2 - H(\nu | \mu) - \chi)]. \quad (3.5)$$

Using this, we get by taking  $\chi = \varepsilon/2$

$$\begin{aligned} &\mathbb{P} \left( M_N(U) \leq e^{N(\log 2 - H(\nu | \mu) - \varepsilon)} \right) \\ &= \mathbb{P} \left( M_N(U) - \mathbb{E} M_N(U) \leq e^{-N\varepsilon/2} e^{N(\log 2 - H(\nu | \mu) - \varepsilon/2)} - \mathbb{E} M_N(U) \right) \\ &\leq \mathbb{P} \left( M_N(U) - \mathbb{E} M_N(U) \leq \left( e^{-N\varepsilon/2} - 1 \right) \mathbb{E} M_N(U) \right) \\ &\leq \mathbb{P} \left( M_N(U) - \mathbb{E} M_N(U) \leq -\frac{1}{2} \mathbb{E} M_N(U) \right) \\ &\leq \mathbb{P} \left( |M_N(U) - \mathbb{E} M_N(U)| \geq \frac{1}{2} \mathbb{E} M_N(U) \right) \\ &\leq 4 \frac{\text{var} [M_N(U)]}{\{ \mathbb{E} M_N(U) \}^2} \leq 4e^{-2N\delta} \leq e^{-\delta N}. \end{aligned}$$

So it remains to prove (3.4). We first claim that for any  $j$

$$\begin{aligned} & \liminf_{r \rightarrow 0} \inf_{\rho, \theta \in \text{cl} B_r(\nu): \rho^{(j)} = \theta^{(j)}} \left\{ H(\rho \mid \mu) + H(\theta \mid \theta^{(j)} \otimes \hat{\mu}^{(j)}) \right\} \\ &= H(\nu \mid \mu) + H(\nu \mid \nu^{(j)} \otimes \hat{\mu}^{(j)}), \end{aligned} \quad (3.6)$$

where  $\hat{\mu}^{(j)} \stackrel{\text{def}}{=} \mu_{j+1} \otimes \cdots \otimes \mu_n$ . The inequality  $\leq$  is evident by taking  $\rho = \theta = \nu$ , and the opposite follows from the semicontinuity properties: One gets that for a sequence  $(\rho_n, \theta_n)$  with  $\rho_n^{(j)} = \theta_n^{(j)}$  and  $\rho_n, \theta_n \rightarrow \nu$ , we have

$$\begin{aligned} & \liminf_{n \rightarrow \infty} H(\rho_n \mid \mu) \geq H(\nu \mid \mu), \\ & \liminf_{n \rightarrow \infty} H(\theta_n \mid \theta_n^{(j)} \otimes \hat{\mu}^{(j)}) \geq H(\nu \mid \nu^{(j)} \otimes \hat{\mu}^{(j)}), \end{aligned}$$

the first inequality by the standard semi-continuity, and the second by Lemma 3.1. This proves (3.6).

Choose  $\eta > 0$  such that  $H(\nu^{(j)} \mid \mu^{(j)}) < \Gamma_j \log 2 - \eta$ , for all  $1 \leq j \leq n$ . By (3.6) we may choose  $r$  small enough such that  $\text{cl} B_r(\nu) \subset V$ , and for all  $1 \leq j \leq n$ ,

$$\begin{aligned} & \inf_{\rho, \theta \in \text{cl} B_r(\nu): \rho^{(j)} = \theta^{(j)}} \left\{ H(\rho \mid \mu) + H(\theta \mid \theta^{(j)} \otimes \hat{\mu}^{(j)}) \right\} \\ & \geq H(\nu \mid \mu) + H(\nu \mid \nu^{(j)} \otimes \hat{\mu}^{(j)}) - \eta/2 \\ & = 2H(\nu \mid \mu) - H(\nu^{(j)} \mid \mu^{(j)}) - \eta/2 \\ & \geq 2H(\nu \mid \mu) - \Gamma_j \log 2 + \eta/2. \end{aligned}$$

For two indices  $\alpha, \alpha'$  we write  $q(\alpha, \alpha') \stackrel{\text{def}}{=} \max \{j : \alpha^{(j)} = \alpha'^{(j)}\}$  with  $\max \emptyset \stackrel{\text{def}}{=} 0$ . Then

$$\begin{aligned} \mathbb{E} M_N^2(U) &= \sum_{j=0}^n \sum_{\alpha, \alpha': q(\alpha, \alpha')=j} \mathbb{P}[L_{N, \alpha} \in U, L_{N, \alpha'} \in U] \\ &= \sum_{\alpha, \alpha': q(\alpha, \alpha')=0} \mathbb{P}[L_{N, \alpha} \in U] \mathbb{P}[L_{N, \alpha'} \in U] \\ &+ \sum_{j=1}^n \sum_{\alpha, \alpha': q(\alpha, \alpha')=j} \mathbb{P}[L_{N, \alpha} \in U, L_{N, \alpha'} \in U] \\ &\leq \mathbb{E}[M_N(\text{cl} U)]^2 + \\ &+ \sum_{j=1}^n \sum_{\alpha, \alpha': q(\alpha, \alpha')=j} \mathbb{P}[L_{N, \alpha} \in \text{cl} U, L_{N, \alpha'} \in \text{cl} U]. \end{aligned}$$

We write the empirical measure in the form (3.3), and use Lemma 3.2. For any  $1 \leq j \leq n$  we have

$$\begin{aligned} & \sum_{\alpha, \alpha': q(\alpha, \alpha')=j} \mathbb{P}[L_{N, \alpha} \in \text{cl} U, L_{N, \alpha'} \in \text{cl} U] \\ &= 2^{\Gamma_j N} 2^{(1-\Gamma_j)N} \left( 2^{(1-\Gamma_j)N} - 1 \right) \mathbb{P}[L_{N, \alpha} \in \text{cl} U, L_{N, \alpha'} \in \text{cl} U], \end{aligned}$$



where on the right hand side  $\alpha, \alpha'$  is an arbitrary pair with  $q(\alpha, \alpha') = j$ . Using Lemma 3.2 we have

$$\begin{aligned}
& \mathbb{P}[L_{N,\alpha} \in \text{cl } U, L_{N,\alpha} \in \text{cl } U] \\
& \leq \exp \left[ -N \inf_{\rho, \theta \in \text{cl } U, \rho^{(j)} = \theta^{(j)}} \left\{ H(\rho^{(j)} | \mu^{(j)}) + \right. \right. \\
& \quad \left. \left. + H(\rho | \rho^{(j)} \otimes \hat{\mu}^{(j)}) + H(\theta | \theta^{(j)} \otimes \hat{\mu}^{(j)}) \right\} + \frac{N\eta}{4} \right] \\
& = \exp \left[ -N \inf_{\rho, \theta \in \text{cl } U, \rho^{(j)} = \theta^{(j)}} \left\{ H(\rho | \mu) + H(\theta | \theta^{(j)} \otimes \hat{\mu}^{(j)}) \right\} + \frac{N\eta}{4} \right] \\
& \leq 2^{\Gamma_j N} \exp \left[ -2NH(\nu | \mu) - \frac{N\eta}{4} \right],
\end{aligned}$$

and thus

$$\sum_{\alpha, \alpha': q(\alpha, \alpha') = j} \mathbb{P}[L_{N,\alpha} \in \text{cl } U, L_{N,\alpha} \in \text{cl } U] \leq 2^{2N} \exp \left[ -2NH(\nu | \mu) - \frac{N\eta}{4} \right].$$

Combining, we obtain by taking  $\chi = \eta/16$  in (3.5)

$$\text{var}[M_N(U)] \leq 2^{2N} \exp \left[ -2NH(\nu | \mu) - \frac{N\eta}{4} \right] \leq e^{-N\eta/8} \mathbb{E}[M_N(U)]^2,$$

which proves our claim.  $\square$

*Proof of Theorem 2.3.* We set

$$\mathcal{G} \stackrel{\text{def}}{=} \left\{ \nu \in \mathcal{M}_1^+(S^n) : H(\nu^{(j)} | \mu^{(j)}) \leq \Gamma_j \log 2, j = 1, \dots, n \right\},$$

which is a compact set.

*Step 1.* We first prove the lower bound. By compactness of  $\mathcal{G}$  and the semicontinuity of  $H$  there exists  $\nu_0 \in \mathcal{G}$  such that

$$\sup_{\nu \in \mathcal{G}} \{\Phi(\nu) - H(\nu | \mu)\} = \Phi(\nu_0) - H(\nu_0 | \mu).$$

We set  $\nu_\lambda \stackrel{\text{def}}{=} (1 - \lambda)\nu_0 + \lambda\mu$  for  $0 < \lambda < 1$ . By convexity of  $H(\nu | \mu)$  in  $\nu$  we see that  $H(\nu_\lambda^{(j)} | \mu^{(j)}) < \Gamma_j \log 2$  for all  $1 \leq j \leq n$ . Furthermore  $\nu_\lambda \rightarrow \nu_0$  weakly as  $\lambda \rightarrow 0$ , and  $\Phi(\nu_\lambda) \rightarrow \Phi(\nu_0)$ ,  $H(\nu_\lambda | \mu) \rightarrow H(\nu_0 | \mu)$ .

Given  $\varepsilon > 0$  we choose  $\lambda > 0$  such that

$$\Phi(\nu_\lambda) - H(\nu_\lambda | \mu) \geq \Phi(\nu_0) - H(\nu_0 | \mu) - \varepsilon.$$

By the continuity of  $\Phi$  and Lemma 3.5 we find a neighborhood  $U$  of  $\nu_\lambda$ , and  $\delta > 0$  such that

$$\Phi(\theta) - \Phi(\nu_\lambda) \leq \varepsilon, \theta \in U,$$

and

$$\mathbb{P}[M_N(U) \leq 2^N \exp[-NH(\nu_\lambda | \mu) - N\varepsilon]] \leq e^{-\delta N},$$

Then, with probability greater than  $1 - e^{-\delta N}$ ,

$$\begin{aligned} Z_N &= 2^{-N} \sum_{\alpha} \exp [N\Phi(L_{N,\alpha})] \\ &\geq 2^{-N} \sum_{\alpha: L_{N,\alpha} \in U} \exp [N\Phi(L_{N,\alpha})] \\ &\geq \exp [N\Phi(\nu_\lambda) - N\varepsilon] \exp [-NH(\nu_\lambda | \mu) - N\varepsilon] \\ &\geq \exp \left[ N \sup_{\nu \in \mathcal{G}} \{ \Phi(\nu) - H(\nu | \mu) \} - 3N\varepsilon \right]. \end{aligned}$$

By Borel-Cantelli, we therefore get, as  $\varepsilon$  is arbitrary,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log Z_N \geq \sup_{\nu \in \mathcal{G}} \{ \Phi(\nu) - H(\nu | \mu) \}$$

almost surely.

*Step 2.* We prove the upper bound. Let again  $\varepsilon > 0$  and set

$$\overline{\mathcal{G}} \stackrel{\text{def}}{=} \{ \nu : H(\nu | \mu) \leq \log 2 \}.$$

If  $\nu \in \mathcal{G}$  we choose  $r_\nu > 0$  such that  $|\Phi(\theta) - \Phi(\nu)| \leq \varepsilon$ ,  $\theta \in B_{r_\nu}(\nu)$  and

$$\mathbb{P} [M_N(B_{r_\nu}(\nu)) \geq 2^N \exp [-NH(\nu | \mu) + N\varepsilon]] \leq e^{-N\delta_\nu},$$

for some  $\delta_\nu > 0$  and large enough  $N$  (using Lemma 3.3). If  $\nu \in \overline{\mathcal{G}} \setminus \mathcal{G}$  we choose  $r_\nu$  such that  $|\Phi(\theta) - \Phi(\nu)| \leq \varepsilon$ ,  $\theta \in B_{r_\nu}(\nu)$ , and

$$\mathbb{P} [M_N(B_{r_\nu}(\nu)) \neq 0] \leq e^{-N\delta_\nu}, \quad (3.7)$$

again for large enough  $N$  (and by Lemma 3.4). As  $\overline{\mathcal{G}}$  is compact, we can cover it by a finite union of such balls, i.e.

$$\overline{\mathcal{G}} \subset U \stackrel{\text{def}}{=} \bigcup_{j=1}^m B_{r_j}(\nu_j),$$

where  $r_j \stackrel{\text{def}}{=} r_{\nu_j}$ . We also set  $\delta \stackrel{\text{def}}{=} \min_j \delta_{\nu_j}$ . We then estimate

$$Z_N \leq 2^{-N} \sum_{l=1}^m \sum_{\alpha: L_{N,\alpha} \in B_{r_l}(\nu_l)} \exp [N\Phi(L_{N,\alpha})] + 2^{-N} \sum_{\alpha: L_{N,\alpha} \notin U} \exp [N\Phi(L_{N,\alpha})]. \quad (3.8)$$

we first claim that almost surely the second summand vanishes provided  $N$  is large enough, i.e. that there is no  $\alpha$  with  $L_{N,\alpha} \notin U$ . By Sanov's theorem, we have

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P} [L_{N,\alpha} \notin U] \leq -\inf_{\nu \notin U} H(\nu | \mu) < -\log 2.$$

Therefore, almost surely, there is no  $\alpha$  with  $L_{N,\alpha} \notin U$ , and therefore the second summand in (3.8) vanishes for large enough  $N$ , almost surely. The same applies to those summands in the

first part for which  $\nu_l \notin \mathcal{G}$ , using (3.7). We therefore have, almost surely, for large enough  $N$ ,

$$\begin{aligned} Z_N &\leq 2^{-N} \sum_{l: \nu_l \in \mathcal{G}} \sum_{\alpha: L_{N,\alpha} \in B_{r_l}(\nu_l)} \exp [N\Phi(L_{N,\alpha})] \\ &\leq e^{N\varepsilon} \sum_{l: \nu_l \in \mathcal{G}} \exp [N\Phi(\nu_l)] M_N(B_{r_l}(\nu_l)) \\ &\leq e^{2N\varepsilon} \sum_{l: \nu_l \in \mathcal{G}} \exp [N\Phi(\nu_l)] \exp [-NH(\nu_l | \mu)] \\ &\leq e^{2N\varepsilon} m \exp \left[ N \sup_{\nu \in \mathcal{G}} \{ \Phi(\nu) - H(\nu | \mu) \} \right]. \end{aligned}$$

As  $\varepsilon$  is arbitrary, we get

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log Z_N \leq \sup_{\nu \in \mathcal{G}} [\Phi(\nu) - H(\nu | \mu)].$$

This finishes the proof of Theorem 2.3.  $\square$

**3.2. The dual representation. Proof of the Theorem 2.5.** We define a family  $\mathcal{G}(\phi) = \{G_{\phi, \mathbf{m}}\}$  of probability distributions on  $S^n$  which depend on the parameter  $\mathbf{m} = (m_1, \dots, m_n) \in \Delta$ . The probability measure  $G = G_{\phi, \mathbf{m}}$  is described by a “starting” measure  $\gamma$  on  $S$ , and for  $2 \leq j \leq n$  Markov kernels  $K_j$  from  $S^{j-1}$  to  $S$ , so that  $G$  is the semi-direct product

$$\begin{aligned} G &= \gamma \otimes K_2 \otimes \dots \otimes K_n. \\ \gamma(dx) &\stackrel{\text{def}}{=} \frac{\exp [m_1 \phi_1(x)] \mu_1(dx)}{\exp [m_1 \phi_0]}, \\ K_j(\mathbf{x}^{(j-1)}, dx_j) &\stackrel{\text{def}}{=} \frac{\exp [m_j \phi_j(\mathbf{x}^{(j)})] \mu_j(dx_j)}{\exp [m_j \phi_{j-1}(\mathbf{x}^{(j-1)})]}, \end{aligned}$$

where we write  $\mathbf{x}^{(j)} \stackrel{\text{def}}{=} (x_j, \dots, x_j)$ . Remember the definition of the function  $\phi_j : S^j \rightarrow \mathbb{R}$  in (2.4), (2.5). It should be remarked that these objects are defined for all  $\mathbf{m} \in \mathbb{R}^n$ , and not just for  $\mathbf{m} \in \Delta$ . We also write

$$G^{(j)} \stackrel{\text{def}}{=} \gamma \otimes K_2 \otimes \dots \otimes K_j$$

which is the marginal of  $G$  on  $S^j$ . In order to emphasize the dependence on  $\mathbf{m}$ , we occasionally will write  $\phi_{j, \mathbf{m}}$ ,  $\gamma_{\mathbf{m}}$ ,  $K_{j, \mathbf{m}}$  etc.

We remark that by a simple computation

$$\begin{aligned} &\int H(K_j(\mathbf{x}^{(j-1)}, \cdot) | \mu_j) G^{(j-1)}(d\mathbf{x}^{(j-1)}) \\ &= m_j \left[ \int \phi_j dG^{(j)} - \int \phi_{j-1} dG^{(j-1)} \right]. \end{aligned} \tag{3.9}$$

$\phi_j, \dots, \phi_n$  do not depend on  $m_j$ , but  $\phi_0, \dots, \phi_{j-1}$  do. Differentiating the equation

$$e^{m_{r+1}\phi_r} = \int e^{m_{r+1}\phi_{r+1}} d\mu_{r+1}$$

with respect to  $m_j$ , we get for  $0 \leq r \leq j-2$

$$\frac{\partial \phi_r(\mathbf{x}^{(r)})}{\partial m_j} = \int \frac{\partial \phi_{r+1}(\mathbf{x}^{(r)}, x_{r+1})}{\partial m_j} K_{r+1}(d\mathbf{x}^{(r)}, x_{r+1}), \tag{3.10}$$

and for  $r = j - 1$

$$\phi_{j-1} e^{m_j \phi_j} + m_j \frac{\partial \phi_{j-1}}{\partial m_j} e^{m_j \phi_j} = \int \phi_j e^{m_j \phi_j} d\mu_j,$$

i.e.

$$\frac{\partial \phi_{j-1}}{\partial m_j} (\mathbf{x}^{(j)}) = \frac{1}{m_j} \left[ \int \phi_j (\mathbf{x}^{(j-1)}, x_j) K_j (\mathbf{x}^{(j-1)}, dx_j) - \phi_{j-1} (\mathbf{x}^{(j-1)}) \right].$$

Combining that with (3.9), (3.10) we get

$$\begin{aligned} \frac{\partial \phi_{0,\mathbf{m}}}{\partial m_j} &= \frac{1}{m_j} \left[ \int \phi_j dG^{(j)} - \int \phi_{j-1} dG^{(j-1)} \right] \\ &= \frac{1}{m_j^2} \int H \left( K_j (\mathbf{x}^{(j-1)}, \cdot) \mid \mu_j \right) G^{(j-1)} (d\mathbf{x}^{(j-1)}). \end{aligned} \quad (3.11)$$

Theorem 2.5 is immediate from the following result:

**Proposition 3.6.** *Assume that  $\phi : S^n \rightarrow \mathbb{R}$  is bounded and continuous. Then there is a unique measure  $\nu$  maximizing Gibbs  $(\nu, \phi)$  under the constraint  $\nu \in \bigcap_{j=1}^n \mathcal{R}_j$ . This measure is of the form  $\nu = G_{\phi, \mathbf{m}}$  where  $\mathbf{m}$  is the unique element in  $\Delta$  minimizing (2.7). For this  $\mathbf{m}$ , we have*

$$\text{Gibbs}(G, \phi) = \text{Parisi}(\phi, \mathbf{m}). \quad (3.12)$$

*Proof.* From strict convexity of the relative entropy, and the fact that  $\bigcap_{j=1}^n \mathcal{R}_j$  is compact and convex, it follows that there is a unique maximizer  $\nu$  of Gibbs  $(\nu, \phi)$  under this constraint.

Also, a straightforward application of Hölder's inequality shows that Parisi  $(\phi, \mathbf{m})$  is a strictly convex function in the variables  $1/m_j$ . Therefore, it follows that there is a uniquely attained minimum of Parisi  $(\phi, \mathbf{m})$  as a function of  $\mathbf{m} \in \Delta$ . This minimizing  $\mathbf{m} = (m_1, \dots, m_n)$ , we can be split into subblocks of equal values: There is a number  $K$ ,  $0 \leq K \leq n$ , and indices  $0 < j_1 < j_2 < \dots < j_K \leq n$  such that

$$\begin{aligned} 0 < m_1 &= \dots = m_{j_1} < m_{j_1+1} = \dots = m_{j_2} \\ &< m_{j_2+1} \dots < m_{j_{K-1}+1} = \dots = m_{j_K} \\ &< m_{j_K+1} = \dots = m_n = 1. \end{aligned}$$

$K = 0$  just means that all  $m_i = 1$ . If  $j_K = n$ , then all  $m_i$  are  $< 1$ . We write  $G = G_{\phi, \mathbf{m}}$ .

From (3.11), we immediately have

$$\frac{\partial \text{Parisi}(\phi, \mathbf{m})}{\partial m_j} = \frac{1}{m_j^2} \left[ \int H \left( K_j (\mathbf{x}^{(j-1)}, \cdot) \mid \mu_j \right) G^{(j-1)} (d\mathbf{x}^{(j-1)}) - \gamma_j \log 2 \right]. \quad (3.13)$$

Set  $d_j \stackrel{\text{def}}{=} \int H \left( K_j (\mathbf{x}^{(j-1)}, \cdot) \mid \mu_j \right) G_{\mathbf{m}}^{(j-1)} (d\mathbf{x}^{(j-1)})$ . We use (3.13) and the minimality of Parisi  $(\phi, \cdot)$  at  $\mathbf{m}$ . We can perturb  $\mathbf{m}$  by moving a whole block  $m_{j_r+1} = \dots = m_{j_{r+1}}$  up and down locally, without leaving  $\Delta$ , provided it is not the possibly present block of values 1. This leads to

$$\sum_{i=j_r+1}^{j_{r+1}} d_i = \log 2 \sum_{i=j_r+1}^{j_{r+1}} \gamma_i.$$

Furthermore, we can always move first parts of blocks, say  $m_{j_r+1} = \dots = m_k$ ,  $k \leq j_{r+1}$  locally down, without leaving  $\Delta$ , so that we get

$$\sum_{i=j_r+1}^{j_k} d_i \leq \log 2 \sum_{i=j_r+1}^{j_k} \gamma_i.$$

These two observations imply

$$G \in \bigcap_{j=1}^n \mathcal{R}_j \cap \bigcap_{r=1}^K \mathcal{R}_{j_r}^{\equiv}. \quad (3.14)$$

We next prove

$$\text{Gibbs}(\nu, \phi) \leq \text{Gibbs}(G, \phi) \quad (3.15)$$

for any  $\nu \in \bigcap_{j=1}^n \mathcal{R}_j$ .

We first prove the case  $n = 1$ . If  $m < 1$ , then

$$H(G | \mu) = \log 2 \geq H(\nu | \mu)$$

by (3.14) and the assumption  $\nu \in \mathcal{R}_1$ . Therefore, in any case

$$\begin{aligned} \text{Gibbs}(G, \phi) - \text{Gibbs}(\nu, \phi) &\geq \int \phi dG - \frac{1}{m} H(G | \mu) \\ &\quad - \left[ \int \phi d\nu - \frac{1}{m} H(\nu | \mu) \right] \\ &= \frac{1}{m} H(\nu | G) \geq 0 \end{aligned}$$

The general case follows by a slight extension of the above argument. Put

$$D_k \stackrel{\text{def}}{=} \int \phi_k dG^{(k)} - \frac{1}{m_{k+1}} H(G^{(k)} | \mu^{(k)}) - \int \phi_k d\nu^{(k)} + \frac{1}{m_{k+1}} H(\nu^{(k)} | \mu^{(k)}),$$

$D_0 \stackrel{\text{def}}{=} 0$ ,  $D_n = \text{Gibbs}(G, \phi) - \text{Gibbs}(\nu, \phi)$ . We prove  $D_{k-1} \leq D_k$  for all  $k$ , so that the claim follows. Remark that as above in the  $n = 1$  case, if  $m_k < m_{k+1}$ , then  $H(G^{(k+1)} | \mu^{(k+1)}) = \Gamma_k \log 2$ , and therefore, in any case

$$\begin{aligned} D_k &\geq \int \phi_k dG^{(k)} - \frac{1}{m_k} H(G^{(k)} | \mu^{(k)}) - \int \phi_k d\nu^{(k)} + \frac{1}{m_k} H(\nu^{(k)} | \mu^{(k)}) \\ &= \int \phi_{k-1} dG^{(k-1)} - \frac{1}{m_k} H(G^{(k-1)} | \mu^{(k-1)}) - \int \phi_k d\nu^{(k)} + \frac{1}{m_k} H(\nu^{(k)} | \mu^{(k)}). \end{aligned}$$

As

$$\begin{aligned} &H(\nu^{(k)} | \mu^{(k)}) - m_k \int \phi_k d\nu^{(k)} + m_k \int \phi_{k-1} d\nu^{(k-1)} \\ &= H(\nu^{(k-1)} | \mu^{(k-1)}) + \int \log \frac{\nu^{(k)}(dx_k | \mathbf{x}^{(k-1)}) e^{m_k \phi_{k-1}(\mathbf{x}^{(k-1)})}}{\mu_k(dx_k) e^{m_k \phi_k(\mathbf{x}^{(k)})}} \nu^{(k)}(d\mathbf{x}^{(k)}) \\ &\geq H(\nu^{(k-1)} | \mu^{(k-1)}), \end{aligned}$$

(3.15) is proved.

(3.14) and (3.15) identify  $G = G_{\phi, \mathbf{m}}$  as the unique maximizer of  $G(\cdot, \phi)$  under the constraint  $\bigcap_{j=1}^n \mathcal{R}_j$ .

The identification (3.12) comes by a straightforward computation.  $\square$

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